

ON INFINITE DIMENSIONAL ALGEBRAIC TRANSFORMATION GROUPS

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To E. B. Dynkin on his 90th anniversary

ABSTRACT. We explore orbits, rational invariant functions, and quotients of the natural actions of connected, not necessarily finite dimensional subgroups of the automorphism groups of irreducible algebraic varieties. The applications of the results obtained are given.

1. Introduction. The following well-known result (see, e.g., [Bor 91, Prop. I.2.2]) is one of the indispensable tools in the theory of algebraic groups:

Theorem. *Let $\varphi_i: T_i \rightarrow G$ ($i \in \mathcal{I}$) be a collection of morphisms from irreducible algebraic varieties T_i into an algebraic group G , and assume that the identity element of G lies in $X_i := \varphi_i(T_i)$ for each $i \in \mathcal{I}$. Then the subgroup A of G generated, as an abstract group, by the set $M := \bigcup_{i \in \mathcal{I}} X_i$ coincides with the intersection of all closed subgroups of G containing M . Moreover, A is connected and there is a finite sequence $(\alpha_1, \dots, \alpha_n)$ in \mathcal{I} such that $A = X_{\alpha_1}^{e_1} \cdots X_{\alpha_n}^{e_n}$, where $e_i = \pm 1$ for each i .*

Here we show that the analogous construction, applied in place of G to $\text{Aut}(X)$, where X is an irreducible algebraic variety, yields a group, though not in general algebraic, but whose natural action on X surprisingly retains some basic properties of orbits and fields of invariant rational functions for algebraic group actions. This leads to some applications.

In general, the groups $\text{Aut}(X)$ are infinite dimensional. Endowing them with the structures of infinite dimensional algebraic groups goes back to [Sha 66], [Sha 82], where this is done for $X = \mathbf{A}^n$ (in modern terminology, the affine Cremona group $\text{Aut}(\mathbf{A}^n)$ is the ind-group). A modification of the argument from [Sha 82] shows that $\text{Aut}(X)$ is actually an ind-group for any affine X .

In [MO 67] a functorial approach to $\text{Aut}(X)$ was developed.

The important concept of algebraic family in $\text{Aut}(X)$ was introduced and elaborated in [Ram 64]; this led to the notions of a connected subgroup and an infinite dimensional subgroup of $\text{Aut}(X)$. Later in [Ser 10] the same idea was embodied in the definition of the Zariski topology of $\text{Aut}(X)$ and $\text{Bir}(X)$ (see Remark 1 below).

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In [Ram 64] it was for the first time discovered that infinite dimensional connected subgroups of $\text{Aut}(X)$ retain some properties of finite dimensional ones, namely, that orbits are open in their closure.

A simple method of constructing many one-dimensional unipotent subgroups of $\text{Aut}(X)$ by means of a single such subgroup U was described in [Pop 87] (in [AFKKZ 13] they are called replicas of U) and applied to constructing non-triangular actions of \mathbf{G}_a on \mathbf{A}^n . This method was then used in the proof of a statement, still existing as folklore (see Appendix), that ensures infinite dimensionality of $\text{Aut}(X)$ in many cases.

In [Pop 05, Defs. 2.1 and 2.2] attention was drawn to considering the subgroups of $\text{Aut}(X)$ (in general, infinite dimensional) generated by one-dimensional unipotent subgroups of $\text{Aut}(X)$. They were then applied in [Pop 11] to constructing a big stock of varieties with trivial Makar-Limanov invariant. Later this topic was developed further in [AFKKZ 13], where the subgroup of $\text{Aut}(X)$ generated by all one-dimensional unipotent subgroups of $\text{Aut}(X)$ was considered. If these one-dimensional subgroups act on X “in all directions”, in [AFKKZ 13], using replicas, the geometric manifestation of infinite dimensionality of this subgroup — infinite transitivity of its action on X — was proved. In [AFKKZ 13] another property was found retained under passing from finite dimensional groups to some infinite dimensional ones: it was proved that the analogue of classical Rosenlicht’s theorem about the existence of rational quotient holds for any subgroup of $\text{Aut}(X)$ generated by a collection of finite dimensional connected algebraic subgroups.

In the present paper we show that actually the analogue of classical Rosenlicht’s theorem holds true for *every* connected subgroup G of $\text{Aut}(X)$. The proof is heavily based on another result proved in this paper: loosely speaking, it claims that the action of G on X is in a sense “reduced” to the “action” of a finite dimensional family in $\text{Aut}(X)$. The applications of these results concern, in particular, the topic of multiple transitivity of the actions on X of connected subgroups of $\text{Aut}(X)$; we show that it is intimately related to unirationality of X . We demonstrate how this can be applied to proving unirationality of some varieties, e.g., the Calogero–Moser spaces and the varieties of n -dimensional representations of a fixed representation type of a finitely generated free associative algebra. The precise formulations of the main results are given in Section 3, and the necessary definitions are collected in Section 2.

In what follows, variety means algebraic variety in the sense of Serre over an algebraically closed field k of arbitrary characteristic (so algebraic group means algebraic group over k). The standard notation and conventions of [Bo91] and [PV94] are used freely. Given a rational function $f \in k(X)$ and an element $\sigma \in \text{Aut}(X)$, we denote by f^σ the rational function on X defined by $f^\sigma(\sigma(x)) = f(x)$ for every point x in the domain of definition of f .

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2. Definitions and notation. Let T be an irreducible variety. Any map

$$\varphi: T \rightarrow \text{Aut}(X), \quad t \mapsto \varphi_t,$$

determines a *family* $\{\varphi_t\}_{t \in T}$ in $\text{Aut}(X)$ parameterized by T . We put

$$\varphi_T := \varphi(T).$$

If \mathcal{I} is a nonempty collection of families in $\text{Aut}(X)$, then the subgroup of $\text{Aut}(X)$ generated, as an abstract group, by the set $\bigcup \varphi_T$ with the union taken over all families $\{\varphi_t\}_{t \in T}$ in \mathcal{I} will be called the *group generated by \mathcal{I}* .

We shall say that a family $\{\varphi_t\}_{t \in T}$ in $\text{Aut}(X)$ is

- *injective* (see [Ram 64]) if $\varphi_t \neq \varphi_s$ for all $t \neq s$;
- *unital* if $\text{id}_X \in \varphi_T$;
- *algebraic* (see [Ram 64]) if

$$\tilde{\varphi}: T \times X \rightarrow X, \quad (t, x) \mapsto \varphi_t(x) \quad (1)$$

is a morphism.

If $\{\varphi_t\}_{t \in T}$ is an algebraic family in $\text{Aut}(X)$ and $\tau: S \rightarrow T$ a morphism, then $\{\psi_s := \varphi_{\tau(s)}\}_{s \in S}$ is also an algebraic family in $\text{Aut}(X)$. If τ is surjective, then $\varphi_T = \psi_S$. Since S may be taken smooth and τ surjective (even such that $\dim S = \dim T$ and τ is proper [Jon 96]), every subgroup of $\text{Aut}(X)$ generated by a collection of unital algebraic families in $\text{Aut}(X)$ is also generated by a collection of unital algebraic families $\{\varphi_t\}_{t \in T}$ with smooth T .

Given a family $\{\varphi_t\}_{t \in T}$ in $\text{Aut}(X)$, the family $\{\varphi_t^{-1}\}_{t \in T}$ in $\text{Aut}(X)$ will be called the *inverse* of $\{\varphi_t\}_{t \in T}$. If $\{\varphi_t\}_{t \in T}, \dots, \{\psi_s\}_{s \in S}$ is a finite sequence of families in $\text{Aut}(X)$, the family

$$\{\varphi_t \circ \dots \circ \psi_s\}_{(t, \dots, s) \in T \times \dots \times S} \quad (2)$$

in X will be called the *product* of $\{\varphi_t\}_{t \in T}, \dots, \{\psi_s\}_{s \in S}$. The inverses and products of families contained in a subgroup G of $\text{Aut}(X)$ are contained in G as well. The inverses and products of algebraic (resp., unital) families are algebraic; see [Ram 64] (resp., unital).

Let \mathcal{I} be a collection of families in $\text{Aut}(X)$. We shall say that a *family* $\{\varphi_t\}_{t \in T}$ in $\text{Aut}(X)$ is *derived from \mathcal{I}* if $\{\varphi_t\}_{t \in T}$ is a product of families each of which is either a family from \mathcal{I} or the inverse of such a family.

A subgroup G of $\text{Aut}(X)$ is called (see [Ram 64]) a *finite dimensional subgroup* if there is an integer n such that $\dim T \leq n$ for every injective algebraic family $\{\varphi_t\}_{t \in T}$ in this subgroup; the smallest n satisfying this property is called the *dimension of G* . If G is not finite dimensional, it is called an *infinite dimensional subgroup* of $\text{Aut}(X)$.

If for every element $g \in G$ there exists a unital algebraic family $\{\varphi_t\}_{t \in T}$ in G such that $g \in \varphi_T$, then G is called (see [Ram 64]) a *connected subgroup* of $\text{Aut}(X)$.

If $\{\varphi_t\}_{t \in T}$ is an algebraic family such that T is a connected algebraic group and $\tilde{\varphi}$ (given by (1)) is an action of T on X , then φ_T is a connected finite dimensional subgroup of $\text{Aut}(X)$. By [Ram 64, Thm.], every connected finite dimensional subgroup of $\text{Aut}(X)$ is obtained in this way. Such subgroups are called *connected algebraic subgroups of $\text{Aut}(X)$* .

Given a nonempty subset S of $\text{Aut}(X)$, we put

$$S(x) := \{g(x) \mid g \in S\}.$$

Given a subgroup G of $\text{Aut}(X)$ and a G -invariant subset Y of X , we shall say that a family $\{\varphi_t\}_{t \in T}$ in G is an *exhaustive family for the natural action of G on Y* if $G(y) = \varphi_T(y)$ for every point $y \in Y$.

Remark 1. [Ram 64] and this paper demonstrate the fruitfulness of the idea of considering specific families. Another example of its embodiment is obtained by replacing $\text{Aut}(X)$ by $\text{Bir}(X)$ and algebraic families by rational ones (i.e., such that $\tilde{\varphi}$ is a rational map): e.g., using such families, J.-P. Serre defines the important notion of the Zariski topology on the Cremona groups [Ser 10]. One can expect fruitfulness of its implementation in other categories (holomorphic families, differentiable families, ...).

3. Main results. In Lemma 1, Theorems 1, 2, 3 and Corollaries 1, 2 below, we do *not* assume finite dimensionality of G . If G is finite dimensional, then the statement of Theorem 1 becomes trivial and that of Theorems 2, 3 and Corollaries 1, 2 turn into the well-known classical results of the algebraic transformation group theory (see, e.g., [PV 94, Sect. 1.4, 2.3]); in particular, Theorem 3 becomes classical Rosenlicht's theorem [Ros 56].

The proofs of the following statements are given in the next sections.

Lemma 1. *Let X be an irreducible variety and let G be a subgroup of $\text{Aut}(X)$. Then the following properties are equivalent:*

- (i) *G is a connected subgroup of $\text{Aut}(X)$;*
- (ii) *G is generated by a collection \mathcal{I} of unital algebraic families in $\text{Aut}(X)$.*

The proof is given in Section 4.

Theorem 1. *Let X be an irreducible variety and let G be a subgroup of $\text{Aut}(X)$ generated by a collection \mathcal{I} of unital algebraic families in $\text{Aut}(X)$. Let Y be a G -invariant locally closed subvariety of X . Then there is a family derived from \mathcal{I} and exhaustive for the natural action of G on Y .*

The proof is given in Section 6.

Orbits of connected subgroups of $\text{Aut}(X)$ are locally closed subvarieties of X (see below, Lemma 4), so one can speak about their dimension.

Theorem 2. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Let Y be an irreducible G -invariant locally closed subvariety of X . Then there exists an integer $m_{G,Y}$ and a dense open subset U of Y such that $\dim G(y) = m_{G,Y}$ for every point $y \in U$.*

The proof is given in Section 7.

Theorem 3. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Let Y be an irreducible G -invariant locally closed subvariety of X . Then for some G -invariant dense open subset U of Y there exists a geometric quotient, i.e., there are an irreducible variety Z and a morphism $\rho: U \rightarrow Z$ such that*

- (i) *ρ is surjective, open, and the fibers of ρ are the G -orbits in U ;*

(ii) if V is an open subset of U , then

$$\rho^*: k[\rho(V)] \rightarrow \{f \in k[V] \mid f \text{ is constant on the fibers of } \rho|_V\}$$

is an isomorphism of k -algebras.

The proof is given in Section 8.

Corollary 1. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Let Y be an irreducible G -invariant locally closed subvariety of X . Then there exists a finite subset of $k(Y)^G$ that separates G -orbits of points of a dense open subset of Y .*

Corollary 2. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Let Y be an irreducible G -invariant locally closed subvariety of X . Then the transcendence degree of the field $k(Y)^G$ over k is equal to $\dim X - m_{G,Y}$ (see Theorem 2). In particular, $k(Y)^G = k$ if and only if there is an open G -orbit in Y .*

Here are some applications of these results.

Theorem 4. *Let X be a nonunirational irreducible variety. Then there exists a nonconstant rational function on X which is G -invariant for every connected affine algebraic subgroup G of $\text{Aut}(X)$.*

The proof is given in Section 10.

Theorem 4 shows that there is a certain rigidity for the orbits of any connected affine algebraic group G acting regularly on an irreducible nonunirational variety X : every such orbit should lie in a level variety of a certain nonconstant rational function on X not depending on G or on its action on X .

Remark 2. “Nonunirational” in Theorem 4 can not be replaced by “nonrational”. Indeed, by [Pop 13, Thm. 2] there exist a connected linear algebraic group G and its finite subgroup F such that $X := G/F$ is not stably rational; since the natural action of G on X is transitive, $k(X)^G = k$.

We shall say that $\text{Aut}(X)$ is *generically n -transitive* if there exists a dense open subset X_n of X such that for every point $x, y \in (X_n)^n$ lying off the union of the “diagonals”, there exists an element $g \in \text{Aut}(X)$ such that $g(x) = y$ for the diagonal action of $\text{Aut}(X)$ on X^n .

In the literature there are many examples of generically n -transitive varieties with $n \geq 2$; see [Rei 93], [KZ 99], [Pop 07], [AFKKZ 13], [BEE 14]. Unirationality of these varieties is proved in many cases (see, e.g., [AFKKZ 13, Prop. 5.1]) and no examples of nonunirational varieties of this type are known. The following Theorem 5 and Corollary 3 concern this topic and make it more likely that such examples do not exist; in the proof we shall assume that k is uncountable, e.g., $k = \mathbf{C}$.

Theorem 5. *Let X be an irreducible variety such that $\text{Aut}(X)$ is generically 2-transitive. Then at least one of the following holds:*

- (i) X is unirational;
- (ii) $\text{Aut}(X)$ contains no nontrivial connected algebraic subgroups.

The proof is given in Section 11.

In fact, I have no examples of X such that $\text{Aut}(X)$ is generically 2-transitive and contains no nontrivial algebraic subgroups.

Corollary 3. *Let X be an irreducible complete variety. If $\text{Aut}(X)$ is generically 2-transitive, then X is unirational.*

The proof is given in Section 12.

As applications of Theorem 5, we obtain the following Corollaries 4 and 5:

Corollary 4. *Every Calogero–Moser space*

$$\mathcal{C}_n := \{(A, B) \in \text{Mat}_n(\mathbf{C})^2 \mid \text{rk}([A, B] + I_n) = 1\} // \text{PGL}_n(\mathbf{C})$$

(see [Wil 98]) *is an irreducible unirational variety.*

The proof is given in Section 13; it is based on Theorem 5 and multiple transitivity of $\text{Aut}(\mathcal{C}_n)$. Using other special properties of \mathcal{C}_n , one can prove that \mathcal{C}_n is actually rational; see Remark 5 in Section 13.

Corollary 5. *For $\text{char } k = 0$ and $m \geq 3$, every set $Q_{m,n}(\tau)$ of all points of $\text{Mat}_n(k)^m // \text{PGL}_n(k)$ of a fixed representation type τ (see [Rei 93]) is an irreducible unirational variety.*

The proof is given in Section 14; it is based on Theorem 5 and the multiple transitivity of $\text{Aut}(Q_{m,n}(\tau))$.

Question 1. Is $Q_{m,n}(\tau)$ rational?

Other applications are discussed in Section 10.

4. Proof of Lemma 1. (i) \Rightarrow (ii): For every element $g \in G$, fix a unital algebraic family $\{\varphi_t\}_{t \in T}$ in G such that $g \in \varphi_T$; the connectedness of G implies that such a family exists. Then G is generated, as an abstract group, by $\bigcup \varphi_T$ with the union taken over all the fixed families.

(ii) \Rightarrow (i): Since the inverse of any family in G is also a family in G , we may (and shall) assume that if a family belongs to \mathcal{I} , then its inverse belongs to \mathcal{I} too. Then for every element $g \in G$, there exists a finite sequence of families $\{\varphi_t\}_{t \in T}, \dots, \{\psi_s\}_{s \in S}$ from \mathcal{I} such that $g = \varphi_{t_0} \circ \dots \circ \psi_{s_0}$ for some $t_0 \in T, \dots, s_0 \in S$. Hence g is contained in the product of families $\{\varphi_t\}_{t \in T}, \dots, \{\psi_s\}_{s \in S}$ defined by (2). Therefore, G is connected. \square

5. Algebraic families. This section contains several general facts utilized in the proofs of Theorems 1, 2, and 3.

Lemma 2. *Let X be an irreducible variety, let G be a connected subgroup of $\text{Aut}(X)$, and let Y be a G -invariant locally closed subvariety of X .*

- (i) *Every product of unital families in $\text{Aut}(X)$ contains each of them.*
- (ii) *If a family $\{\varphi_t\}_{t \in T}$ in G is exhaustive for the natural action of G on Y , then every family $\{\psi_s\}_{s \in S}$ in G such that $\varphi_T \subseteq \psi_S$ is also exhaustive for this action.*
- (iii) *If G is generated by a collection \mathcal{I} of unital algebraic families, then G is the union of all families derived from \mathcal{I} .*

- (iv) $G|_Y := \{g|_Y \mid g \in G\}$ is a connected subgroup of $\text{Aut}(Y)$.
- (v) If \mathcal{F} is a finite set of algebraic families in G , then G contains a unital algebraic family $\{\varphi_t\}_{t \in T}$ such that $\varphi_T \supseteq \psi_S$ for every $\{\psi_s\}_{s \in S}$ in \mathcal{F} .

Proof. (i) and (ii): This is immediate from the definitions.

(iii): The proof is similar to that of implication (ii) \Rightarrow (i) of Lemma 1.

(iv): If $\{\varphi_t\}_{t \in T}$ is a unital algebraic family in G containing an element $g \in G$, then $\{\varphi_t|_Y\}_{t \in T}$ is a unital algebraic family in $G|_Y$ containing the element $g|_Y \in G|_Y$. Whence the claim.

(v): Due to (i), the proof is reduced to the case where \mathcal{F} consists of a single family $\{\psi_s\}_{s \in S}$. In this case, take an element $g \in \psi_S$. Since G is connected, it contains a unital algebraic family $\{\mu_r\}_{r \in R}$ such that $g^{-1} \in \mu_R$. The product of $\{\psi_s\}_{s \in S}$ and $\{\mu_r\}_{r \in R}$ is then the sought-for family $\{\varphi_t\}_{t \in T}$. \square

Lemma 3. *Let X be an irreducible variety and let Y be a locally closed subvariety of X . Let Y_1, \dots, Y_n be all the irreducible components of Y . If $\{\varphi_t\}_{t \in T}$ is a unital algebraic family in $\text{Aut}(X)$ such that Y is φ_t -invariant for every $t \in T$, then every Y_i is φ_t -invariant for every $t \in T$.*

Proof. For every point $t \in T$, since $\varphi_t \in \text{Aut}(X)$ and Y is φ_t -invariant, φ_t permutes Y_1, \dots, Y_n . Put

$$T_{ij} := \{t \in T \mid \varphi_t(Y_i) = Y_j\}.$$

For every point $x \in Y_i$ consider the morphism

$$\tilde{\varphi}_x: T \rightarrow X, \quad t \mapsto \tilde{\varphi}(t, x) = \varphi_t(x) \quad (3)$$

(see (1)). Then, for every Y_j ,

$$T_{ij} = \bigcap_{x \in Y_i} \tilde{\varphi}_x^{-1}(Y_j). \quad (4)$$

Since Y_j is closed, (4) implies the closedness of T_{ij} in T . Unitality of φ_t implies $T_{ii} \neq \emptyset$. From $T = \bigsqcup_{j=1}^n T_{ij}$ and the irreducibility of T we then infer that $T = T_{ii}$ for every i , i.e., Y_i is φ_t -invariant for every i and t . \square

Lemma 3 and the definition of connected subgroups of $\text{Aut}(X)$ yield

Corollary 6. *Let X , Y , and Y_1, \dots, Y_n be the same as in Lemma 3. If Y is G -invariant for a connected subgroup G of $\text{Aut}(X)$, then every Y_i is G -invariant.*

Lemma 4. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. If $\{\varphi_t\}_{t \in T}$ is an algebraic family in G , and x is a point of X , then*

- (i) $G(x)$ is an irreducible locally closed nonsingular subvariety of X ;
- (ii) $\varphi_T(x)$ is a constructible subset of $G(x)$.

Proof. (i): This is proved in [Ram 64, Lemma 2].

(ii): This follows from the definition of algebraic family and Chevalley's theorem on the image of morphism. \square

Corollary 7. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Then $k(X)^G$ is algebraically closed in $k(X)$.*

Proof. Let $f \in k(X)$ be a root of $t^n + f_1 t^{n-1} + \dots + f_n \in k(X)^G[t]$ and let $a \in X$ be a point where f and every f_i are defined. Then by Lemma 4(i) the restriction of f to the irreducible variety $G(a)$ is a well-defined rational function $f|_{G(a)} \in k(G(a))$. The image of the rational map $f|_{G(a)} : G(a) \dashrightarrow k$ is a finite set since it lies in the set of roots of $t^n + f_1(a)t^{n-1} + \dots + f_n(a) \in k[t]$. Irreducibility of $G(a)$ then implies that this image is a single element of k , i.e., $f|_{G(a)}$ is a constant. Whence $f \in k(X)^G$. \square

Lemma 5. *Let X be an irreducible variety and let G be a connected subgroup of $\text{Aut}(X)$. Let Y be a G -invariant locally closed subvariety of X . Let $\{\varphi_t\}_{t \in T}$ be an algebraic family in G such that $\varphi_T(y)$ is dense in $G(y)$ for every point $y \in Y$. Then the product of the inverse of $\{\varphi_t\}_{t \in T}$ and $\{\varphi_t\}_{t \in T}$ is the unital algebraic family $\{\psi_s\}_{s \in S}$ in G exhaustive for the natural action of G on Y .*

Proof. By the definition of $\{\psi_s\}_{s \in S}$,

$$\psi_s = \varphi_{t_1}^{-1} \circ \varphi_{t_2} \quad \text{for } s = (t_1, t_2) \in S = T \times T. \quad (5)$$

Take any points $y_1, y_2 \in Y$ such that $\overline{G(y_1)} = \overline{G(y_2)}$. The density assumption then yields the equality $\overline{\varphi_T(y_1)} = \overline{\varphi_T(y_2)}$, where bar stands for the closure in X . By Lemma 4, this implies

$$\varphi_T(y_1) \cap \varphi_T(y_2) \neq \emptyset;$$

whence, $\varphi_{t_1}(y_2) = \varphi_{t_2}(y_1)$ for some $t_1, t_2 \in T$. Therefore, $\psi_s(y_1) = y_2$ for ψ_s defined by (5). Hence $\psi_S(y_1) = G(y_1)$ for every point $y_1 \in Y$, i.e., $\{\psi_s\}_{s \in S}$ is exhaustive for the action of G on Y . Its unitality follows from (5). \square

6. Proof of Theorem 1. First, we shall show that it suffices to prove the following “generic” version of Theorem 1:

Theorem 1*. *Let X , G , \mathcal{I} , and Y are the same as in Theorem 1 and let Y be irreducible. Then there exist a dense open G -invariant subset U in Y and a unital algebraic family $\{\varphi_t\}_{t \in T}$ in G such that*

- (i) $\{\varphi_t\}_{t \in T}$ is derived from \mathcal{I} ;
- (ii) $\varphi_T(y)$ is dense in $G(y)$ for every point $y \in U$.

Indeed, assuming that Theorem 1* is proved, we can complete the proof of Theorem 1 as follows.

The group G is connected by Lemma 1. Therefore, every irreducible component of Y is G -invariant by Corollary 6. From this and Lemma 2(i),(ii) we infer that it is sufficient to prove Theorem 1 for irreducible Y . In this case we argue by induction on $\dim Y$.

Namely, the case $\dim Y = 0$ is clear. Assume that the claim of Theorem 1 holds for irreducible G -invariant subvarieties in X of dimension $< \dim Y$ and consider the set U from Theorem 1*. Let Z_1, \dots, Z_n be all the irreducible components of the variety $Y \setminus U$. By Corollary 6, every Z_i is G -invariant. Since $\dim Z_i < \dim Y$, the inductive assumption implies for every $i = 1, \dots, n$ the existence of a unital algebraic family $\{\psi_{s_i}^{(i)}\}_{s_i \in S_i}$ in G such that

- (a) $\{\psi_{s_i}^{(i)}\}_{s_i \in S_i}$ is derived from \mathcal{I} ;
- (b) $\{\psi_{s_i}^{(i)}\}_{s_i \in S_i}$ is exhaustive for the natural action of G on Z_i .

On the other hand, Theorem 1* and Lemma 5 imply the existence of a unital algebraic family $\{\lambda_r\}_{r \in R}$ in G such that

- (c) $\{\lambda_r\}_{r \in R}$ is derived from \mathcal{I} ;
- (d) $\{\lambda_r\}_{r \in R}$ is exhaustive for the natural action of G on U .

The claim of Theorem 1 now follows from (a), (b), (c), (d) and Lemma 2(i),(ii). This completes the proof of Theorem 1 assuming that Theorem 1* is proved. \square

We now turn to the proof of Theorem 1*. Consider the map

$$\tau_Y: G \times Y \rightarrow Y \times Y, \quad (g, y) \mapsto (g(y), y). \quad (6)$$

Its image Γ_Y is the graph of the natural action of G on Y :

$$\Gamma_Y = \{(y_1, y_2) \in Y \times Y \mid G(y_1) = G(y_2)\}. \quad (7)$$

Claim 1. *Maintain the above notation.*

- (i) *There exists a family $\{\varphi_t\}_{t \in T}$ derived from \mathcal{I} such that $\tau_Y(\varphi_T \times Y)$ contains a dense open subset V of $\overline{\Gamma_Y}$, where bar stands for the closure in $Y \times Y$.*
- (ii) *$\overline{\Gamma_Y}$ is irreducible.*

Proof of Claim 1. If $\{\psi_s\}_{s \in S}$ is an algebraic family in G , then the subset $\tau_Y(\psi_S \times Y)$ of Γ_Y is the image of the morphism

$$S \times Y \rightarrow Y \times Y, \quad (s, y) \mapsto (\psi_s(y), y)$$

of irreducible varieties (see (1)). Chevalley's theorem on the image of morphism then implies that $\overline{\tau_Y(\psi_S \times Y)}$ is an irreducible subvariety of $\overline{\Gamma_Y}$ and $\tau_Y(\psi_S \times Y)$ contains a dense open subset of $\overline{\tau_Y(\psi_S \times Y)}$.

From $\dim \overline{\Gamma_Y} \geq \dim \overline{\tau_Y(\psi_S \times Y)}$ we conclude that there exists a family $\{\varphi_t\}_{t \in T}$ derived from \mathcal{I} on which the maximum of $\dim \overline{\tau_Y(\psi_S \times Y)}$ is attained when $\{\psi_s\}_{s \in S}$ runs over all families derived from \mathcal{I} . If $\{\psi_s\}_{s \in S}$ is a family derived from \mathcal{I} such that $\varphi_T \subseteq \psi_S$, then the maximality condition and irreducibility of $\overline{\tau_Y(\psi_S \times Y)}$ imply that

$$\overline{\tau_Y(\psi_S \times Y)} = \overline{\tau_Y(\varphi_T \times Y)}. \quad (8)$$

Take an element $g \in G$. By Lemma 2(iii),(i), there is an algebraic family $\{\psi_s\}_{s \in S}$ in G such that $\varphi_T \subseteq \psi_S$ and $g \in \psi_S$. From (8) and (6) we then conclude that $\Gamma_Y \subseteq \overline{\tau_Y(\varphi_T \times Y)}$. Since $\overline{\tau_Y(\varphi_T \times Y)} \subseteq \overline{\Gamma_Y}$, we get $\overline{\tau_Y(\varphi_T \times Y)} = \overline{\Gamma_Y}$. This completes the proof. \square

Endow $X \times X$ with the action of G via the second factor:

$$g \cdot (x_1, x_2) := (x_1, g(x_2)), \quad x_i \in X, g \in G. \quad (9)$$

The second projection $X \times X \rightarrow X$, $(x_1, x_2) \mapsto x_2$ is then G -equivariant and, by (7), Γ_Y and $\overline{\Gamma_Y}$ are G -invariant.

Claim 2. *$\{\varphi_t\}_{t \in T}$ and V in Claim 1 can be chosen so that V is G -invariant.*

Proof of Claim 2. Maintain the notation of Claim 1 and consider in $\overline{\Gamma}_Y$ the G -invariant dense open subset

$$V_0 := \bigcup_{g \in G} g \cdot V. \quad (10)$$

Since V_0 is quasi-compact, its covering (10) by open subsets $g \cdot V$, $g \in G$, contains a finite subcovering:

$$V_0 = \bigcup_i^n g_i \cdot V \quad \text{for some elements } g_1, \dots, g_n \in G. \quad (11)$$

By Lemma 2(iii), every g_i is contained in a family derived from \mathcal{I} . Taking a product of $\{\varphi_t\}_{t \in T}$ with these families, we obtain a family $\{\psi_s\}_{s \in S}$ derived from \mathcal{I} such that

$$\varphi_T \circ g_i^{-1} \subseteq \psi_S \quad \text{for every } i = 1, \dots, n. \quad (12)$$

Since $V \subseteq \tau_Y(\varphi_T \times Y)$, from (6) and (9) we obtain

$$g_i \cdot V \subseteq \{(\varphi_t(y), g_i(y)) \mid t \in T, y \in Y\}. \quad (13)$$

This yields

$$\begin{aligned} \tau_Y(\psi_S \times Y) &= \{(\psi_s(y), y) \mid s \in S, y \in Y\} \\ &= \{(\psi_s(g_i(y)), g_i(y)) \mid s \in S, y \in Y\} \\ &\supseteq \{(\varphi_t(g_i^{-1}(g_i(y))), g_i(y)) \mid t \in T, y \in Y\} \quad (\text{by (12)}) \\ &\supseteq g_i \cdot V \quad (\text{by (13)}). \end{aligned} \quad (14)$$

Thus $V_0 \subseteq \tau_Y(\psi_S \times Y)$ by (11) and (14). So, replacing $\{\varphi_t\}_{t \in T}$ and V by, resp., $\{\psi_s\}_{s \in S}$ and V_0 , we may attain that V in Claim 1 is G -invariant. \square

To complete the proof of Theorem 1*, consider the second projection

$$\pi_Y : \overline{\Gamma}_Y \rightarrow Y, \quad (y_1, y_2) \mapsto y_2; \quad (15)$$

it is a G -equivariant surjective morphism of irreducible varieties. Let $\{\varphi_t\}_{t \in T}$ and V be as in Claim 1 and let V be G -invariant by Claim 2. Since V is a dense open subset of $\overline{\Gamma}_Y$, by Chevalley's theorem on the image of morphism, $\pi_Y(V)$ contains a dense open subset of Y . Let U be the union of all dense open subsets of Y lying in $\pi_Y(V)$. Since V is G -invariant and π_Y is G -equivariant, $\pi_Y(V)$ is G -invariant. Therefore, U is also G -invariant.

Take a point $y \in U$. Since $V \subseteq \Gamma_Y$, $\pi_Y^{-1}(y) \cap \Gamma_Y = \{(g(y), y) \mid g \in G\}$, and $V \supseteq \{(g(y), y) \mid g \in \varphi_T\}$, we have

$$\emptyset \neq V \cap \pi_Y^{-1}(y) = V \cap \Gamma_Y \cap \pi_Y^{-1}(y) = V \cap \{(g(y), y) \mid g \in G\} \quad (16)$$

$$\subseteq \{(g(y), y) \mid g \in \varphi_T\}. \quad (17)$$

By Lemma 4, $\{(g(y), y) \mid g \in G\}$ is an irreducible locally closed subset of $\overline{\Gamma}_Y$. From (16) we then infer that $V \cap \{(g(y), y) \mid g \in G\}$ is a dense open subset of $\{(g(y), y) \mid g \in G\}$, and from (17) that $\varphi_T(y)$ is dense in $G(y)$. This completes the proof of Theorem 1* and hence that of Theorem 1. \square

7. Proof of Theorem 2. Maintain the notation of the proof of Theorem 1. It is proved there that the restriction of π_Y to V is a dominant morphism of irreducible varieties $V \rightarrow Y$ whose fiber over every point y of a dense open subset U of Y is isomorphic to a dense subvariety of $G(y)$. Hence, the dimension of this fiber is $\dim G(y)$. The claim now follows from the fiber dimension theorem [Gro 65, 5.6]. \square

8. Proof of Theorem 3. By Lemma 2(iv), it suffices to give a proof for $Y = X$. We shall use the idea utilized in [Lun 73, 4] for proving the existence of a generic stabilizer for reductive group actions on smooth affine varieties. Below is maintained the notation used in the proof of Theorem 1.

The plan is to repeat several times the procedure of replacing X by its open dense subset having some necessary additional properties; in order to avoid unnecessary extra notation, this subset will still be denoted by X . An open subset of the original X obtained at the last step will be the sought-for U from the formulation of Theorem 3.

Since any subfield of $k(X)$ containing k is finitely generated over k , replacing X by an appropriate invariant dense open subset of X we can (and shall) find an irreducible affine normal variety Z and a surjective morphism

$$\rho: X \rightarrow Z$$

such that $\rho^*(k(Z)) = k(X)^G$. This equality implies that ρ is a separable morphism; see, e.g., [Bor 91, AG, Prop. 2.4].

The construction yields that

$$(q_1) \quad G(x) \subseteq \rho^{-1}(\rho(x)) \text{ for every point } x \in X.$$

By the fibre dimension theorem and Theorem 2, further replacing X and Z by the appropriate open sets, we can (and shall) attain the following properties:

- (q₂) for every point $z \in Z$, the dimension of every irreducible component of $\rho^{-1}(z)$ is equal to $\dim X - \dim Z$;
- (q₃) $\dim G(x) = \dim G(x')$ for all points $x, x' \in X$.

Lemma 4(i) and (q₃) imply that $G(x)$ is closed in X for every point $x \in X$.

By Grothendieck's generic freeness lemma [Gro 65, 6.9.2], after replacing Z by a principal open subset, we can (and shall) assume that

- (q₄) there exists an affine open subset X_0 of X such that $\rho(X_0) = Z$ and $k[X_0]$ is a free $\rho^*(k[Z])$ -module.

Below, for any subsets $S \subseteq X$ and $R \subseteq X \times X$, we put

$$S_0 := S \cap X_0, \quad R_0 := R \cap (X_0 \times X_0).$$

Finally, replacing X by the invariant open set $\bigcup_{g \in G} g(X_0)$, we can (and shall) assume that

- (q₅) the intersection of X_0 with every G -orbit in X is nonempty.

Consider now in $X \times X$ the G -invariant (with respect to action (9)) closed subset

$$X \times_Z X := \{(x_1, x_2) \in X \times X \mid \rho(x_1) = \rho(x_2)\} \quad (18)$$

and its affine open subset $(X \times_Z X)_0$.

Claim 3. $(X \times_Z X)_0$ is dense in $X \times_Z X$.

Proof of Claim 3. Take a point $(x_1, x_2) \in X \times_Z X$. From (18) and (q₁) we infer that $G(x_1) \times G(x_2) \subseteq X \times_Z X$, and from (q₅) and Lemma 4(i) that $(G(x_1) \times G(x_2))_0$ is a dense open subset of $G(x_1) \times G(x_2)$. Therefore, since $(x_1, x_2) \in G(x_1) \times G(x_2)$, the closure of $(G(x_1) \times G(x_2))_0$ in $X \times_Z X$ contains (x_1, x_2) . Whence the claim, because $(G(x_1) \times G(x_2))_0 \subseteq (X \times_Z X)_0$. \square

Next, consider the set

$$\Gamma := \Gamma_X \quad (19)$$

defined by (7). By (q₁), we have $\Gamma \subseteq X \times_Z X$. Since $X \times_Z X$ is closed in $X \times X$, this yields $\bar{\Gamma} \subseteq X \times_Z X$ (see Claim 1(i)).

Claim 4. $\bar{\Gamma} = X \times_Z X$.

First, we shall show how to deduce Theorem 3 from Claim 4.

By (19) and Claims 1(ii), 4, the variety $\bar{\Gamma} = X \times_Z X$ is irreducible. Consider its dense open subset V from Claim 2 and morphism $\pi_X: \bar{\Gamma} \rightarrow X$ defined by (15) for $Y = X$. If B is an irreducible component of $\bar{\Gamma} \setminus V$ such that $\pi_X(B)$ is dense in X , then, by the fiber dimension theorem, $\dim \pi_X^{-1}(x) > \dim \pi_X^{-1}(x) \cap B$ for every point $x \in X$ lying off a proper closed subset of X . This and property (q₃) imply that $V \cap \pi_X^{-1}(x)$ is dense in $\pi_X^{-1}(x)$ for every such x . On the other hand, $\pi_X^{-1}(x) = \rho^{-1}(\rho(x)) \times x$ by (18) and, as explained at the end of the proof of Theorem 1, $V \cap \pi_X^{-1}(x)$ is a dense open subset of $G(x) \times x$. Since $G(x) \subseteq \rho^{-1}(\rho(x))$, this shows that $G(x)$ is dense in $\rho^{-1}(\rho(x))$. The closedness of $G(x)$ in X then implies that $G(x) = \rho^{-1}(\rho(x))$ for every point $x \in X$ lying off a proper closed subset. This means that replacing Z by its open subset and X by the inverse image of this subset, we can (and shall) assume that ρ is an orbit map, i.e., the fibers of ρ are the G -orbits in X . Since ρ is a surjective separable morphism and Z is a normal variety, by [Bor 91, Prop. II.6.6] this implies that $\rho: X \rightarrow Z$ is the geometric quotient. Thus the proof of Theorem 3 is completed provided that Claim 4 is proved. \square

So it remains to prove Claim 4.

Proof of Claim 4. We divide it into three steps.

1. In view of Claim 3, it suffices to prove the density of Γ_0 in $(X \times_Z X)_0$. Since $(X \times_Z X)_0$ is an affine variety, the latter is reduced to proving that if a function $f \in k[(X \times_Z X)_0]$ vanishes on Γ_0 ,

$$f|_{\Gamma_0} = 0, \quad (20)$$

then $f = 0$. To prove this, note that the closedness of $(X \times_Z X)_0$ in $X_0 \times X_0$ implies the existence of a function $h \in k[X_0 \times X_0]$ such that

$$h|_{(X \times_Z X)_0} = f. \quad (21)$$

In turn, since $k[X_0 \times X_0] = p_1^*(k[X_0]) \otimes_k p_2^*(k[X_0])$, where $p_i: X_0 \times X_0 \rightarrow X_0$, $(x_1, x_2) \mapsto x_i$, there are functions $s_1, \dots, s_m, t_1, \dots, t_m \in k[X_0]$ such that

$$h = \sum_{i=1}^m p_1^*(s_i) p_2^*(t_i). \quad (22)$$

2. By an appropriate replacement of h and $s_1, \dots, s_m, t_1, \dots, t_m$ we may obtain that t_1, \dots, t_m are linearly independent over $\rho^*(k[Z])$. Indeed, by property (q₄), there are functions $b_1, \dots, b_r \in k[X_0]$, linearly independent over $\rho^*(k[Z])$, such that

$$t_i = \sum_{j=1}^r c_{ij} b_j \quad \text{for some } c_{ij} \in \rho^*(k[Z]), \quad i = 1, \dots, m. \quad (23)$$

In view of (22) and (23), we have

$$h = \sum_{j=1}^r \left(\sum_{i=1}^m p_1^*(s_i) p_2^*(c_{ij}) \right) p_2^*(b_j). \quad (24)$$

Take a point $x = (x_1, x_2) \in (X \times_Z X)_0$. Since $\rho(x_1) = \rho(x_2)$, we have

$$c_{ij}(x_1) = c_{ij}(x_2) \quad \text{for all } i, j. \quad (25)$$

From (24) and (25) we then obtain

$$\begin{aligned} h(x) &= \sum_{j=1}^r \left(\sum_{i=1}^m s_i(x_1) c_{ij}(x_2) \right) b_j(x_2) \\ &= \sum_{j=1}^r \left(\sum_{i=1}^m s_i(x_1) c_{ij}(x_1) \right) b_j(x_2). \end{aligned} \quad (26)$$

Hence if we put

$$\begin{aligned} d_j &:= \sum_{i=1}^m s_i c_{ij} \in k[X_0], \\ \tilde{h} &:= \sum_{j=1}^r p_1^*(d_j) p_2^*(b_j) \in k[X_0 \times X_0], \end{aligned} \quad (27)$$

then we have $h(x) = \tilde{h}(x)$ by virtue of (26). Given (21), this yields

$$\tilde{h}|_{(X \times_Z X)_0} = f. \quad (28)$$

From (27) and (28) we conclude that the replacement of s_1, \dots, s_m and t_1, \dots, t_m by, respectively, d_1, \dots, d_r and b_1, \dots, b_r is the one we are looking for.

3. Thus, keeping the notation, we shall now assume that t_1, \dots, t_m in (22) are linearly independent over $\rho^*(k[Z])$.

Take an element $g \in G$ and let D be the domain of definition of the rational function

$$\ell = \sum_{i=1}^m s_i t_i^g \in k(X).$$

Since X is irreducible, $D \cap g(D) \cap X_0 \cap g(X_0)$ is a dense open subset of X . Let x be a point of this subset. Then the rational functions $\ell, s_i, t_i^g \in k(X)$ are defined at x and

$$a := (x, g^{-1}(x)) \in \Gamma_0. \quad (29)$$

From this we obtain

$$\begin{aligned} \ell(x) &= \sum_{i=1}^m s_i(x) t_i^g(x) = \sum_{i=1}^m s_i(x) t_i(g^{-1}(x)) \\ &\stackrel{\text{by (29)}}{=} \left(\sum_{i=1}^m p_1^*(s_i) p_2^*(t_i) \right)(a) \stackrel{\text{by (22)}}{=} h(a) \stackrel{\text{by (21)}}{=} f(a) \stackrel{\text{by (20)}}{=} 0. \end{aligned}$$

So ℓ vanishes on a dense open subset of X ; whence $\ell = 0$. Thus, it is proved that

$$\sum_{i=1}^m s_i t_i^g = 0 \quad \text{for every } g \in G. \quad (30)$$

Since Z is affine and $\rho^*(k(Z)) = k(X)^G$, the field of fractions of $\rho^*(k[Z])$ is $k(X)^G$. This implies that t_1, \dots, t_m are linearly independent over $k(X)^G$. In

turn, by Artin's theorem [Bou 59, §7, no. 1, Thm. 1], this linear independency yields the existence of elements $g_1, \dots, g_m \in G$ such that

$$\det(t_i^{g_j}) \neq 0. \quad (31)$$

Combining (30) and (31) we obtain $s_1 = \dots = s_m = 0$. From this, (22), and (21), we then infer that $f = 0$, as claimed. \square

9. Distinguished connected subgroups of $\text{Aut}(X)$. Some collections \mathcal{I} of unital algebraic families in $\text{Aut}(X)$ are naturally distinguished. They generate distinguished connected subgroups $\text{Aut}(X)_{\mathcal{I}}$ of $\text{Aut}(X)$ that are of interest.

The first example is the collection \mathcal{U} of all unital algebraic families in $\text{Aut}(X)$. We shall denote $\text{Aut}(X)_{\mathcal{U}}$ by $\text{Aut}(X)^0$ and call it the *identity component* of $\text{Aut}(X)$. The group $\text{Aut}(X)/\text{Aut}(X)^0$ will be called *the component group* of $\text{Aut}(X)$.

Proposition 1. *Let X be an irreducible variety such that $\text{Aut}(X)$ is a finite group. Then $\text{Aut}(X)^0 = \{\text{id}_X\}$.*

Proof. Let $\{\varphi_t\}_{t \in T}$ be a unital algebraic family in $\text{Aut}(X)$. Take a point $x \in X$. Irreducibility of T implies irreducibility of the image I_x of morphism (3). Finiteness of $\text{Aut}(X)$ (resp., unitality of $\{\varphi_t\}_{t \in T}$) implies finiteness of I_x (resp., $x \in I_x$). This yields $I_x = \{x\}$, i.e., $\varphi_T = \{\text{id}_X\}$; whence the claim. \square

Remark 3. For any finite group G , there is a smooth affine irreducible variety X such that $\text{Aut}(X)$ and G are isomorphic; see [Jel 94].

The component group of $\text{Aut}(X)$, in contrast to that of an algebraic group, may be infinite.

Remark 4. If k is uncountable, then the same argument as in the proof of Proposition 1 shows that if $\text{Aut}(X)$ is countable (such X do exist, see Examples 1, 2 below), then $\text{Aut}(X)^0 = \{\text{id}_X\}$ and hence the component group of $\text{Aut}(X)$ is countable.

Example 1. Let X be a surface in \mathbf{A}^3 defined by the equation $x_1^2 + x_2^2 + x_3^2 = x_1 x_2 x_3 + a$ where $a \in k$. By [Èl-H 74], if a is generic, then $\text{Aut}(X)$ contains a subgroup of finite index which is a free product of three subgroups of order 2.

Example 2. Let $\text{char } k = 0$ and let X be a smooth irreducible quartic in \mathbf{P}^3 . Then $\text{Aut}(X)^0 = \{\text{id}_X\}$ by [Mat 63], and, according to the classical Fano–Severi result, for a sufficiently general X there is a bijection between $\text{Aut}(X)$ and the (countable) set of solutions $(a, b), a > 0$ of the Pell equation $x^2 - 7y^2 = 1$ (see [MM 64, pp. 353–354]).

Example 3. Let X be the underlying variety of an algebraic torus G of dimension $n > 0$. The automorphism group $\text{Aut}_{\text{gr}}(G)$ of the algebraic group G is embedded in $\text{Aut}(X)$ and isomorphic to $\text{GL}_n(\mathbf{Z})$. The map $G \rightarrow \text{Aut}(X)$, $g \mapsto \ell_g$, where $\ell_g: X \rightarrow X$, $x \mapsto gx$, identifies G with a subgroup of $\text{Aut}(X)$. By [Ros 61, Thm. 3],

$$\text{Aut}(X) = \text{Aut}_{\text{gr}}(G) \ltimes G. \quad (32)$$

Let $\{\varphi_t\}_{t \in T}$ be a unital algebraic family in $\text{Aut}(X)$. By [Ros 61, Thm. 2] there are the morphisms $\alpha: T \rightarrow G$ and $\beta: X \rightarrow X$ such that $\varphi_t(x) = \tilde{\varphi}(t, x) = \ell_{\alpha(t)}(\beta(x))$ for every $t \in T$, $x \in X$ (see (1)). Put $s := \beta(e)$. Since $(\ell_{s^{-1}} \circ \beta)(e) = e$, [Ros 61, Thm. 3] implies that $g := \ell_{s^{-1}} \circ \beta \in \text{Aut}_{\text{gr}}(G)$. From $\beta = \ell_s \circ g$ we then infer that $\varphi_t(x) = \ell_{\alpha(t)}(\ell_s(g(x))) = \ell_{\alpha(t)s}(g(x))$; whence $\varphi_t = \ell_{\alpha(t)s} \circ g$. This, (32), and the unitality of $\{\varphi_t\}_{t \in T}$ imply that $g = \{\text{id}_X\}$. Therefore, $\varphi_T \subseteq G$. This proves that $\text{Aut}(X)^0 = G$ and the component group of $\text{Aut}(X)$ is isomorphic to $\text{GL}_n(\mathbf{Z})$.

Example 4. By [Ram 64, Cor. 1], $\text{Aut}(X)^0$ is a connected algebraic group if X is an irreducible complete variety (and, in fact, more generally, semi-complete variety, i.e., if for any torsion free coherent algebraic sheaf \mathcal{F} on X , the k -vector space $H^0(X, \mathcal{F})$ of sections is finite dimensional).

Theorem 6. $\text{Aut}(\mathbf{A}^n) = \text{Aut}(\mathbf{A}^n)^0$ for every n .

Proof. We shall apply the argument going back to [Ale 23] and utilized in [Sha 82, Lemma 4]. Let x_1, \dots, x_n be the standard coordinate functions on \mathbf{A}^n . Any element of $\text{Aut}(X)$ is a composition of an element of the affine group $\text{Aff}_n := \{a \in \text{Aut}(X) \mid \deg a^*(x_i) \leq 1 \text{ for every } i\}$ and an element $g \in \text{Aut}(X)$ such that

$$g^*(x_i) = x_i + \sum_{j=2}^d f_{ij}, \quad i = 1, \dots, n, \quad (33)$$

where every $f_{ij} \in k[x_1, \dots, x_n]$ is either zero or a form of degree j . Given that Aff_n is a connected algebraic group, this reduces the proof to demonstrating that g is contained in a unital algebraic family in $\text{Aut}(X)$.

This can be done as follows. For every $t \in k = \mathbf{A}^1$, $t \neq 0$, define $h_t \in \text{Aff}_n$ by

$$h_t^*(x_i) = tx_i, \quad i = 1, \dots, n, \quad (34)$$

and put $g_t := h_t^{-1} \circ g \circ h_t \in \text{Aut}(X)$. Then (33) and (34) yield

$$g_t^*(x_i) = x_i + \sum_{j=2}^d t^{j-1} f_{ij}, \quad i = 1, \dots, n. \quad (35)$$

Putting $g_0 := \text{id}_{\mathbf{A}^n}$, we deduce from (35) that $\{g_t\}_{t \in \mathbf{A}^1}$ is a unital algebraic family in $\text{Aut}(X)$, and from (33) that $g_1 = g$. This completes the proof. \square

A series of examples is obtained taking \mathcal{I} to be a part of the collection \mathcal{G} of all algebraic families $\{\varphi_t\}_{t \in T}$ such that T is a connected algebraic group and $\tilde{\varphi}$ defined by (1) is an action of T on X . In this case, $\text{Aut}(X)_{\mathcal{I}}$ is a subgroup of $\text{Aut}(X)$ generated, as an abstract group, by a collection of some connected algebraic subgroups of $\text{Aut}(X)$. For $\text{char } k = 0$, the subgroups $\text{Aut}(X)_{\mathcal{I}}$ of this type were studied in [AFKKZ 13, Sect. 1] where they are called “algebraically generated groups of automorphisms”. Propositions 1.3, 1.5 and Theorem 1.13 of [AFKKZ 13] are the special cases of, respectively, the above Lemma 4, Theorem 1, and Theorem 3.

Some interesting parts \mathcal{I} of \mathcal{G} are obtained as collections of all families $\{\varphi_t\}_{t \in T}$ in \mathcal{G} such that the algebraic group T has a certain property.

For instance, requiring that T is affine one obtains the collection \mathcal{G}_{aff} . Theorems 4 and 5 give examples of dependency between the groups $\text{Aut}(X)_{\mathcal{G}}$, $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}}$ and geometric properties of X . Here is another example.

Example 5. If $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}} \neq \{\text{id}_X\}$, then X is birationally isomorphic to the product of \mathbf{A}^1 and a variety of dimension $\dim X - 1$. This follows from [Mat 63, Cor. 1].

Developing the idea of [Pop 11, Def. 1.36], one obtains another example of an interesting collection of families taking \mathcal{I} to be the collection $\mathcal{G}(F)$ of all families $\{\varphi_t\}_{t \in T}$ in \mathcal{G} such that T is isomorphic to a fixed connected algebraic group F .

For $F = \mathbf{G}_a$ this yields the important subgroup $\text{Aut}(X)_{\mathcal{G}(\mathbf{G}_a)}$ in $\text{Aut}(X)$, introduced¹ in [Pop 05, Def. 2.1] and called in this paper “ ∂ -generated subgroup”. Its close relation to constructing a big stock of varieties with trivial Makar-Limanov invariant was shown in [Pop 11]. Later in [AFKKZ 13] the transitivity properties of $\text{Aut}(X)_{\mathcal{G}(\mathbf{G}_a)}$ (called in this paper “the special automorphism group of X ” and denoted by² $\text{SAut}(X)$) were studied. By [Pop 11, Lemma 1.1], $\text{Aut}(X)_{\mathcal{G}(\mathbf{G}_a)}$ coincides with the subgroup of $\text{Aut}(X)$ generated by all connected affine subgroups of $\text{Aut}(X)$ that have no nontrivial characters.

Another interesting case is $F = \mathbf{G}_m$. Since the union of all maximal tori of a connected reductive group is dense in it, $\text{Aut}(X)_{\mathcal{G}(\mathbf{G}_m)}$ coincides with the subgroup of $\text{Aut}(X)$ generated by all connected reductive subgroups of $\text{Aut}(X)$. This implies that

$$\text{Aut}(X)_{\mathcal{G}_{\text{aff}}} = \text{Aut}(X)_{\mathcal{G}(\mathbf{G}_a)} \cup \text{Aut}(X)_{\mathcal{G}(\mathbf{G}_m)}.$$

Indeed, let H be a connected affine algebraic group with a maximal torus T and the unipotent radical $R_u(H)$, and let $\pi: H \rightarrow H/R_u(H)$ be the canonical projection. By [Bor 91, Prop. 11.20], $\pi(T)$ is a maximal torus in $H/R_u(H)$. The conjugacy of maximal tori and the density of their union in $H/R_u(H)$ yield $H/R_u(H) = \pi(S)$ for the subgroup S in H generated by all maximal tori. Whence the claim.

10. Proof of Theorem 4. Since G lies in $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}}$, by Corollary 2 it suffices to show that neither of the $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}}$ -orbits is open in X .

Assume the contrary and let \mathcal{O} be an $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}}$ -orbit open in X . Take a point $x \in \mathcal{O}$. By Theorem 1, a certain family $\{\varphi_t\}_{t \in T}$ derived from \mathcal{G}_{aff} is exhaustive for the action of $\text{Aut}(X)_{\mathcal{G}_{\text{aff}}}$ on X . Then \mathcal{O} is the image of morphism (3). Since \mathcal{O} is open in X , this morphism is dominant. On the other hand, the definitions of derived family and \mathcal{G}_{aff} imply that T is a product of underlying varieties of connected affine algebraic groups. But such underlying varieties are rational (see [Pop 13, Lemma 2] for a four-lines proof; we failed to find an earlier reference for a proof valid in arbitrary characteristic). Hence T is a rational variety. This and the dominance of morphism (3) then imply that X is unirational — a contradiction. \square

¹At the irrelevant assumption $X = \mathbf{A}^n$.

²A hardly felicitous notation, as in the literature SAut denotes entirely different concept — the group of semialgebraic automorphisms, see Y. Z. Flicker, C. Scheiderer, R. Sujatha, *Grothendieck's theorem on non-abelian H^2 and local-global principles*, J. Amer. Math. Soc. **11** (1998), no. 3, 731–750.

11. Proof of Theorem 5. Let X be nonunirational. Assume that $\text{Aut}(X)$ contains a nontrivial connected affine algebraic subgroup C . Then there exists a point $x \in X_2$ such that $X_2 \cap C(x)$ is an irreducible locally closed set of positive dimension. Hence there exists a point $y \in X_2 \cap C(x)$, $y \neq x$. By the condition of 2-transitivity, for every point $z \in X_2$, $z \neq x$, there exists an element $g \in \text{Aut}(X)$ such that $g(x) = x$, $g(z) = y$. This implies that for the subgroup $H := g^{-1} \circ C \circ g$ we have $z \in H(x)$. Therefore, for the connected subgroup G of $\text{Aut}(X)$ generated by all conjugates of C in $\text{Aut}(X)$ we have $X_2 \subseteq G(x)$; whence $G(x)$ is open in X .

From this, arguing as in the proof of Theorem 4, we deduce that X is unirational — a contradiction. Hence $\text{Aut}(X)$ does not contain nontrivial connected affine algebraic subgroups.

Now assume that $\text{Aut}(X)$ contains a nontrivial connected nonaffine algebraic subgroup A . Since, as we proved, there are no nontrivial connected affine algebraic subgroups in A , the structure theorem on algebraic groups [Bar 55], [Ros 56] implies that A is a nontrivial abelian variety. The same argument as above for C then shows that the connected subgroup of $\text{Aut}(X)$ generated by all conjugates of A in $\text{Aut}(X)$ has an orbit \mathcal{O} which is open in X and admits a surjective morphism $Z \rightarrow \mathcal{O}$, where Z is a product of several copies of the underlying variety of A . Since Z is a complete variety, this implies that X is complete as well and $X = \mathcal{O}$.

The completeness of X implies that $\text{Aut}(X)^0$ is a connected algebraic group and $\text{Aut}(X)/\text{Aut}(X)^0$ is at most countable [MO 67]. Since $\text{Aut}(X)^0$ does not contain nontrivial connected algebraic subgroups, the same argument as above yields that $\text{Aut}(X)^0$ is a nontrivial abelian variety acting transitively on X . The commutativity of $\text{Aut}(X)^0$ and the faithfulness of its action on X then imply that the $\text{Aut}(X)^0$ -stabilizer of every point of X is trivial.

Take a point $x \in X_2$ and let $\text{Aut}(X)_x$ be its $\text{Aut}(X)$ -stabilizer. The assumption of generic 2-transitivity of the action of $\text{Aut}(X)$ on X implies that there is an $\text{Aut}(X)_x$ -orbit containing $X_2 \setminus \{x\}$. But this orbit is at most countable since $\text{Aut}(X)_x \cap \text{Aut}(X)^0 = \{e\}$, while $X_2 \setminus \{x\}$, being open in X , is uncountable (e.g., because an affine open subset of $X_2 \setminus \{x\}$ is a branched covering of an affine space by the Noether lemma) — a contradiction. This completes the proof. \square

12. Proof of Corollary 3. Assume that X is nonunirational. Then by Theorem 5 the group $\text{Aut}(X)$ contains no nontrivial connected algebraic subgroups. Since X is complete, this implies that $\text{Aut}(X)$ is at most countable. A contradiction with the assumption of generic 2-transitivity of the action of $\text{Aut}(X)$ on X is then obtained using the same argument as in the end of the proof of Theorem 5. \square

13. Calogero–Moser spaces.

Proof of Corollary 4. According to [Wil 98], \mathcal{C}_n is an irreducible smooth affine variety. By [BEE 14, Thm. 1], the natural action of $\text{Aut}(\mathcal{C}_n)$ on \mathcal{C}_n is 2-transitive. There are nontrivial connected algebraic subgroups in $\text{Aut}(\mathcal{C}_n)$:

for instance, the action of GL_1 on $\{(A, B) \in \mathrm{Mat}_n(\mathbf{C})^2 \mid \mathrm{rk}([A, B] + I_n) = 1\}$ given by $t \cdot (X, Y) := (t^{-1}X, tY)$ descends to \mathcal{C}_n . Unirationality of \mathcal{C}_n then follows from Theorem 5. \square

Remark 5. One can show that \mathcal{C}_n is actually rational. The proof is based on [Wil 98, Prop. 1.10] and goes as follows.³ Endow \mathbf{A}^n with the standard action of the symmetric group S_n and consider the diagonal action of S_n on $\mathbf{A}^n \times \mathbf{A}^n$. It follows from [Wil 98, Prop. 1.10] that \mathcal{C}_n is birationally isomorphic to $(\mathbf{A}^n \times \mathbf{A}^n)/S_n$. By the No-name Lemma (see, e.g., [Pop 13, Lemma 1]), $(\mathbf{A}^n \times \mathbf{A}^n)/S_n$ is birationally isomorphic to $\mathbf{A}^n \times (\mathbf{A}^n/S_n)$. Since \mathbf{A}^n/S_n is isomorphic to \mathbf{A}^n , the claim follows.

14. Proof of Corollary 5. Irreducibility of $Q_{m,n}(\tau)$ is proved in [LBP 87, Thm. II.1.1]. By [Rei 93, Thm. 1.4], for $m \geq 3$ the natural action of group $\mathrm{Aut}(Q_{m,n}(\tau))$ on $Q_{m,n}(\tau)$ is 2-transitive. There are nontrivial connected algebraic subgroups in $\mathrm{Aut}(Q_{m,n}(\tau))$: for instance, the action of GL_1 on $\mathrm{Mat}_n(k)^m$ by scalar multiplication induces a representation type preserving action on $\mathrm{Mat}_n(k)^m // \mathrm{PGL}_n(k)$; see [Rei 93, Prop. 4.1]. The unirationality of $Q_{m,n}(\tau)$ then follows from Theorem 5. \square

15. Appendix. Here is the folklore statement mentioned in the introduction:

Theorem 7. *Let $\mathrm{char} k = 0$. If X is an irreducible affine variety, $\dim X \geq 2$, and $\mathrm{Aut}(X)$ contains a one-dimensional algebraic unipotent subgroup U , then the group $\mathrm{Aut}(X)$ is infinite dimensional.*

Proof. By Rosenlicht's theorem [Ros 56], $\mathrm{tr} \deg_k k(X)^U \geq \dim X - \dim U > 0$. By [PV 94, Thm. 3.3], the unipotency of U yields the equality $\mathrm{tr} \deg_k k[X]^U = \mathrm{tr} \deg_k k(X)^U$. Since X is affine, $U = \{\exp t\partial \mid t \in k\}$, where ∂ is a locally nilpotent derivation of $k[X]$. The claim then follows from the inclusion $\{\exp h\partial \mid h \in k[X]^U\} \subseteq \mathrm{Aut}(X)$. \square

Remark 6. Example 3 shows that “unipotent” in Theorem 7 cannot be dropped.

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